Bell's theorem without inequalities and only two distant observers

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Abstract

A proof of Bell's theorem without inequalities is given by suitably extending a proof of the Bell-Kochen-Specker theorem due to Mermin. This proof is generalized to obtain an inequality-free proof of Bell's theorem using an entangled state of $2n$ qubits (with n odd) shared between two distant observers.

In two recent papers[1,2], Cabello gave a proof of Bell's theorem without inequalities by using a special state of four qubits shared between two distant observers. This improved upon the classic proof of Greenberger, Horne and Zeilinger[3] and Mermin[4] by reducing the number of distant observers from three to two. The purpose of this paper is to describe a variant of Cabello's proof that avoids one of its shortcomings and that can also be generalized to apply to an entangled state of $2n$ qubits (with n odd) shared between two distant observers. The present proof, like Cabello's[2] and several others before it[5], uses a common framework to prove both the Bell-Kochen-Specker (BKS)[6] and Bell[7] theorems. However, while Cabello proceeds backwards from the stronger (Bell) to the weaker (BKS) theorem, we proceed in the opposite direction. Our approach has the advantage that it makes no use of either entanglement or communication in proving the BKS theorem, and invokes these additional elements only to build the bridge proceeding from the BKS to the Bell theorem.

Figure 1 shows a 3 x 3 array of observables pertaining to a pair of qubits that was used by Mermin[8] to prove the BKS theorem. Mermin's proof is based on the elementary observations that: (a) each observable has only the eigenvalues ± 1, (b) the observables in any row or column of the array form a mutually commuting set, and (c) the product of the observables (and hence their eigenvalues) in any row or column is $+1$, with the exception of the last column for which the product is -1. Armed with these facts, Mermin's argument proceeds as follows.

Suppose an experimenter, Alice, who has two qubits in her possession carries out the measurements corresponding to the commuting observables in one of the rows or columns of Mermin's square. The result will be a set of $+1$s and -1s for the measured eigenvalues satisfying the product constraint mentioned earlier. The product constraint can be restated as the "sum" constraint that the total number of -1s for any triad of commuting observables is always even, except for the triad corresponding to the last column for which it is odd. Now if Alice is a "realist" and believes that the eigenvalues she measures merely reflect preexisting properties of the qubits, she would be tempted to assign the value $+1$ or -1 to each of the nine observables in Mermin's square in such a way that all the sum constraints on their values are met. However this is easily seen to be impossible by counting the total number of -1s in the square in two different ways: firstly, by summing over the rows (which leads to an even number) and, secondly, by summing over the columns (which leads to an odd number). This contradiction shows the impossibility of assigning preexisting values (or "elements of reality"[9]) to the qubits and constitutes Mermin's proof of the BKS theorem.

However the above BKS proof has the objectionable feature that an observable is assigned the same value whether it is measured as part of a row or column of observables. This assumption of "noncontextuality" has no empirical basis and, in the opinion of many physicists (including Bell himself[6]), considerably diminishes the force of the BKS theorem[10]. We now show how to rectify this defect and thereby promote Mermin's BKS proof into a proof of Bell's theorem. To do this we enlist the help of a second experimenter, Bob, give him two qubits of his own, and allow him to do everything Alice can at a location far removed from hers. The trick to ensuring that Alice and Bob can jointly prove Bell's theorem is that the four qubits given to them are in the entangled state

$$|\Psi\rangle = 1\sqrt{2}(|00\rangle + |11\rangle)_{13} \otimes 1\sqrt{2}(|00\rangle + |11\rangle)_{24} = 12\left(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle\right), \tag{1}$$

where 0 and 1 denote the basis states of a qubit and the subscripts $1,..,4$ in the middle expression indicate the relative positions of these qubits in the last, expanded form of the state $|\Psi\rangle$. In other words, $|\Psi\rangle$ consists of a pair of identical Bell states, with one member of each pair (qubits 1 and 2) going to Alice and the other members (qubits 3 and 4) going to Bob. It is also assumed that the first and second members of each two particle observable in Fig.1 refer to qubits 1 and 2 for Alice and qubits 3 and 4 for Bob.

The state (1) possesses the interesting property that $O_i^A O_i^B |\Psi\rangle = |\Psi\rangle$ for $i = 1,...,9$, where O_i^A is any one of Alice's nine observables and O_i^B is the same observable for Bob. This property implies that if Alice and Bob measure identical observables on their qubits, they always obtain the same eigenvalues, even if their measurements are carried out at spacelike separations. These correlated outcomes suffice to establish that all of Alice's observables are "elements of reality", and that the same is true of Bob's observables as well. For Bob (or Alice) can use his (or her) measurement of a particular observable to instantly predict the outcome of the other person's measurement of the same observable at a distant location, without disturbing that person's qubits in any way. Note further that either person's ability to predict the value of the other's observable is independent of whether that observable is measured by itself or as part of any commuting triad it happens to be a member of[11]. But this last statement is just the earlier assumption of noncontextuality, now justified on the basis of the correlations in state (1) and the principle of locality, and serves to promote Mermin's BKS proof into a full fledged proof of Bell's theorem.

Cabello's proof[2] differs from ours in that the state (1) is replaced by a direct product of singlets and the nine observables measured by Alice and Bob are not identical. However the most significant difference is that (in Cabello's scheme) Alice and Bob are required to collaborate in measuring five non-local observables, each made up equally of their separate observables. While the measurement of these non-local observables poses no problems for a Bell test, it imposes the unnecessary burden on a BKS test of using communication between observers to achieve its goals.

The present BKS-Bell proof suggests a joint laboratory experiment for verifying the BKS and Bell theorems. However its practical realization is complicated by the fact that each observer needs to be able to measure a sequence of three commuting two-particle observables on his/her qubits. Such "non-demolition" measurements are possible to envision in principle[12], but they are rather challenging to carry out in practice.

The above Bell proof can be generalized to a suitable entangled state of $2n$ qubits (with n odd) shared between two distant observers. Consider the following $(n+2)$ sets of mutually commuting n-qubit observables, where each commuting set is shown on a separate line and the superscripts on the Pauli operators refer to the different qubits:

$$\sigma_x^1\sigma_z^2\sigma_x^3, \sigma_x^2\sigma_z^3\sigma_x^4, ..., \sigma_x^i\sigma_z^{i+1}\sigma_x^{i+2}, ..., \sigma_x^{n-1}\sigma_z^n\sigma_x^1, \sigma_x^n\sigma_z^1\sigma_x^2, \sigma_z^1\sigma_z^2\cdots\sigma_z^n, \quad (2)$$
$$\sigma_x^1, \sigma_z^2, \sigma_x^3, \sigma_x^1\sigma_z^2\sigma_x^3 \quad (3)$$
$$\sigma_x^2, \sigma_z^3, \sigma_x^4, \sigma_x^2\sigma_z^3\sigma_x^4 \quad (4)$$
$$... \quad ... \quad ... \quad ... \quad ... \quad ... \quad (5)$$
$$\sigma_x^n, \sigma_z^1, \sigma_x^2, \sigma_x^n\sigma_z^1\sigma_x^2 \quad (6)$$
$$\sigma_z^1, \sigma_z^2, \cdot, \cdot, \cdot, \sigma_z^n, \sigma_z^1\sigma_z^2\cdots\sigma_z^n. \quad (7)$$

Each line after the first consists of one of the observables in the first line together with all the single particle observables of which it is made up. There are $(n+1)+2n = 3n+1$ distinct observables in all, each of which occurs in exactly two commuting sets. The observables have the further properties that: (a) each has only the eigenvalues ±1, and (b) the product of the observables (and hence their eigenvalues) in any commuting set is $+1$, with the exception of the first set for which this product is -1.

A BKS proof can now be constructed as follows. Suppose Alice is given n qubits and allowed to measure the above observables on them. If Alice is a "realist" and believes that the eigenvalues she measures already preexist in the qubits, she would be tempted to assign the value $+1$ or -1 to each of the observables in such a way that all the product constraints on their values are met. However this is easily seen to be impossible by setting the product of the values of the observables in each line each line of (2)-(7) equal to $+1$ (or -1 for the first line), and taking the product of all the equations so obtained. One finds then that the product of the left sides of all the equations is $+1$ (because each observable occurs twice and contributes a factor of $+1$), whereas the product of their right sides is -1 (as a result of the first equation). This contradiction rules out the possibility of preexisting values for the observables and proves the BKS theorem using the setting provided by a system of n qubits. We now show how to justify the assumption of noncontextuality implicit in this argument and thereby promote it into a proof of Bell's theorem.

Consider the set of n Bell states, $1\sqrt{2}(|00\rangle + |11\rangle)_{11'} \otimes 1\sqrt{2}(|00\rangle + |11\rangle)_{22'} \otimes ... \otimes 1\sqrt{2}(00+11)_{nn'}$, and suppose that the unprimed member of each Bell state is given to Alice and the primed member to Bob. Suppose also that Alice and Bob are each allowed to measure the $3n+1$ observables listed in (2)-(7) on their respective qubits. Then it is not difficult to verify that if Alice and Bob measure identical observables on their respective qubits, they always obtain the same eigenvalues (identical observables for Alice and Bob differ only in that all the unprimed superscripts in one are replaced by their primed counterparts in the other). From this perfect correlation one can argue, as before, that either observer's observables are elements of reality and hence that our BKS proof based on (2)-(7) can be reinterpreted as a Bell proof.

The $n = 3$ case of our BKS proof was given by Mermin[8], who arranged the ten relevant observables[13] at the vertices of a pentagram in such a way that all the commuting observables lay along a common edge. Mermin converted this BKS proof into a Bell proof by assuming that the three qubits were in a correlated GHZ state. The $n = 5$ case of our BKS proof was given by DiVincenzo and Peres[14], who also pointed out that it could be promoted into a Bell proof if the five qubits were assumed to be in a state corresponding to one of the logical codewords of the five-qubit single error-correcting code[15]. The Bell proofs of Mermin and of DiVincenzo and Peres involve three and five separated observers, respectively, whereas our proofs involve only two observers who however share three or five Bell states between themselves.

Our Bell proofs assume that Alice and Bob share perfect Bell states and that their particle detectors are 100 % efficient. If these conditions do not obtain, our proofs lose their "all or nothing" character and can be rescued only by devising inequalities that are satisfied by local realism but violated by quantum mechanics (and experiment). We now exhibit one such inequality. Suppose Alice and Bob share n EPR singlets, with Alice possessing one member of each pair and Bob the other. Consider the operator $B = B_1 B_2 ... B_n$, where $B_i = \sigma^i \cdot \hat{a}(\sigma^{i'} \cdot \hat{b} + \sigma^{i'} \cdot \hat{b}') + \sigma^i \cdot \hat{a}'(\sigma^{i'} \cdot \hat{b} - \sigma^{i'} \cdot \hat{b}')$ is the usual CHSH operator[16] for the $i-$th singlet shared by Alice and Bob; here σ^i and $\sigma^{i'}$ are the spin operators for Alice and Bob's qubits in the $i-$th singlet and $\hat{a}, \hat{a}', \hat{b}, \hat{b}'$ are the appropriate unit vectors that make the expectation value of B_i in a singlet state achieve its maximal value of $2\sqrt{2}$. Then, on taking the expectation value of B in a direct product of n singlets one finds the value of $(2\sqrt{2})^n$, which can be compared with the maximal value of 2^n yielded by local realism. One finds, therefore, that the gulf between quantum mechanics and local realism widens exponentially with the number, n, of singlets considered. This is similar to Mermin's earlier finding[17] for the $n-$particle GHZ state. It should be added that this inequality-type Bell proof does not rest upon a BKS proof, as our earlier proofs did. The same technique of "amplification" used here can be applied to qudits (i.e. higher spin particles) as well, to produce a larger gulf between the predictions of local realism and quantum mechanics.

To conclude, we have presented a heirarchy of joint BKS-Bell proofs based on a special state of $2n$ qubits (in fact, a collection of n identical Bell states) shared equally between two distant observers. Our proofs illustrate the close relationship between the two foundational theorems of quantum mechanics, and particularly demonstrate how the weaker (BKS) theorem can serve as a catalyst in the proof of the stronger one. Our proofs can be generalized in several ways and also suggest some applications to quantum cryptography and quantum state estimation that will be discussed elsewhere.

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[11] A more detailed explanation of this assertion is as follows. When Alice measures a particular observable on her qubits, she collapses them into the two-dimensional subspace associated with a particular eigenvalue ($+1$ or -1) of that observable. The correlations in state (1) then dictate that Bob's qubits collapse into the same two-dimensional subspace of their Hilbert space. If Bob subsequently measures the same observable as Alice, either alone or in combination with any other observables that commute with it, his qubits remain within the selected two-dimensional subspace and he definitely obtains the same eigenvalue as Alice for the common observable measured.

[12] A "non-demolition" measurement on a set of qubits can be carried out by coupling them to ancilliary qubits and carrying out the usual (destructive) measurements on the ancillas. A quantum circuit, consisting of a sequence of one- and two-qubit gates, can be designed to implement any non-demolition measurement. However the practical implementation of the basic two-qubit gate (the "controlled-not" gate) is still in its infancy and so the ability to fabricate a quantum circuit that will carry out a non-demolition measurement is still in doubt. For an alternative approach to the measurement of sequences of commuting two-qubit observables, see C.Simon, M.Zukowski, H.Weinfurter and A.Zeilinger, Phys. Rev. Lett. 85, 1783 (2000).

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$1 \otimes \sigma_z$	$\sigma_z \otimes 1$	$\sigma_z \otimes \sigma_z$
$\sigma_x \otimes 1$	$1 \otimes \sigma_x$	$\sigma_x \otimes \sigma_x$
$\sigma_x \otimes \sigma_z$	$\sigma_z \otimes \sigma_x$	$\sigma_y \otimes \sigma_y$

Fig.1. A 3 x 3 array of observables for a pair of qubits used by Mermin (ref.8) to prove the Bell-Kochen-Specker theorem.